Nonlinear dynamics of finite perturbation: Collapse and revival of spatial patterns

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A full-scale nonlinear stability analysis is performed on a reaction-diffusion system that includes a cubic polynomial source term and Cattaneo's modification of Fick's law of diffusion. This modification incorporates the effect of a small, finite relaxation time of flux at the macroscopic level of the description of the process. While linear stability analysis predicts the decay of small wavelength perturbations on a homogeneous steady state for large reaction time, consideration of finite perturbations leads to a spatiotemporal instability, resulting in an interesting phenomenon of periodic collapse and revival of spatial patterns. This instability is relaxation (time) driven, and the time period is determined by self-sustaining oscillations due to the limit cycle of the underlying dynamics. The nonlinear dynamics of finite perturbations may thus be generically different from what is expected from a linear stability analysis.

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I. INTRODUCTION

The nonlinear dynamics of zero-dimensional [1] and spatially extended systems [2–4] has received wide attention over the last several decades. Their application cover a range of phenomena encompassing physical, chemical, and biological sciences, particularly in the realm of self-organization of systems under far-from-equilibrium condition. Examples [5–10] include, among others, chemical oscillations, formation of spatial stationary or nonstationary patterns, traveling waves, spiral, targets under different physical conditions accessible to experiments. They are amenable to theoretical analysis in terms of nonlinear kinetics or reaction-diffusion models. The first step towards understanding these phenomena at large lies in exploring the nature of stability by following the dynamics of temporal and spatiotemporal perturbations in the close neighborhood of the homogenous steady states in terms of linear analysis [1,2]. However, when the strength of perturbation is finite and large, nonlinear terms of the perturbation series cannot be neglected. The question is how and to what extent these nonlinear contribution of the perturbative terms generically affect the dynamical behavior of the system around the homogeneous steady states. Although this problem has not been adequately addressed in the literature [4,11–20], on a general footing, a number of interesting attempts have been made in several different but related issues. The effect of finite perturbations on dynamical systems has been considered [17] to generalize the traditional concept of Lyapunov exponents, which characterize the average local stability properties to first order by describing the rate at which the small volumes expand or contract in different directions. Dressler and Farmer [17] have developed higher-order Lyapunov exponents to characterize the nonlinear distortions at quadratic, cubic, etc., orders of a Taylor series when the dynamics becomes locally unstable and nonlinearities become too severe. A scale-dependent Lyapunov exponent to measure the degree of chaoticity has been described [18] to understand the effect of finite perturbations in fully developed turbulance. The authors of [19] have explored the effect of noninfinitesimal perturbations in spatially extended chaotic systems in the form of coupled-map lattices to derive the conditions for evolution controlled by nonlinear mechanisms and an approximate expression for nonlinear velocity rather than standard linear velocity obtained by linear analysis. It has also been shown that for dominant nonlinear effects finite-amplitude disturbance can propagate faster than infinitesimal ones in these systems [20]. The importance of the contribution of nonlinear terms of the spatiotemporal perturbation has also been demonstrated in reaction-diffusion systems where additive and/or multiplicative noise induces instability in the dynamics [4,11–16] under appropriate conditions.

Before further elaboration we note that while linear stability analysis is unique by its very nature and is truly applicable in the limit of infinitesimal perturbation, the scope of a nonlinear perturbative analysis, although broad, is at the same time always limited by its nonuniqueness. And second, in general, any finite-order perturbative series is prone to the problem of convergence. To avoid these pitfalls one should either deal with strategies that allow construction of truly convergent perturbative series or take care of nonlinear terms up to all orders. Our analysis in this paper pertains to the second case. We consider a simple reaction-diffusion system [21–26] that includes Cattaneo's modification of Fick's law of diffusion. This modification of diffusive motion may be perceived at a microscopic level as a persistent random walk which provides a wide range of transport from the ballistic limit to the diffusive limit and incorporates a finite speed of propagation for transport of dispersing species. At the macroscopic level this finite speed corresponds to a nonzero small relaxation time which is required for the adjustment of a delayed flux to the concentration gradient. A few years back Horsthemke [24] carried out a linear stability analysis of a general two-variable reaction-diffusion system of this kind to show that the homogeneous steady state of the system can undergo two types of transport-driven instabilities. One type of bifurcation results in a stationary spatial pattern corresponding to Turing instability and the other type occurs in the ballistic regime leading to an oscillating spatial pattern. The one-variable version of the model, on the other

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hand, exhibits oscillatory decay of small wavelength infinitesimal perturbations. In the present paper we confine ourselves to the one-variable model, but undertake a full-scale nonlinear stability analysis where the dynamics of spatiotemporal perturbations on a homogeneous steady state is expected to be determined by a finite relaxation time in addition to the the source term of the reaction kinetics and diffusion coefficient of the dispersing component. The structure of the resulting parabolic reaction-diffusion equation suggests an underlying dissipative character due to an interplay of nonlinearity of the reaction kinetics and the finite relaxation time of the flux in an appropriate parameter region. Specifically we address the following questions.

(i) Can the nonlinear dissipative character of the reaction-Cattaneo equation induce any new spatiotemporal instability, transient or stationary, in the dynamics? The answer to this question lies in exploring the possibility of emergence of new dynamical attractors like the limit cycle around the homogeneous steady state when nonlinear perturbation terms to all orders are taken into consideration.

(ii) How does the degree of excitation of nonlinear perturbation terms affect the transient characteristics of the dynamics around the homogeneous steady state? One expects that depending on the specificity of strong and weak excitations due to the nonlinearity of the reaction kinetics, a finite relaxation time of flux may play a decisive role in controlling the dynamics of the spatiotemporal perturbations.

(iii) To what extent is the spatiotemporal instability, if any, induced by the relaxation effect in contrast to diffusion? An answer to this question is important for understanding the time scale of the evolution of nonlinear perturbations. Modeling dispersing and reacting systems with a finite relaxation time of flux and a reaction term characterized by a cubic polynomial, we have carried out a nonlinear stability analysis of the homogeneous steady state of the dynamical system. In what follows we show that a finite spatiotemporal perturbation on an unstable steady state neither grows nor decays as observed in linear analyses, but gives rise to generically spatiotemporal structures: e.g., periodic collapse and revival of spatial patterns with a time period given by the limit cycle of self-excited oscillations. We examine both strong and weak excitation limits of the nonlinear perturbation to derive the time period of oscillations. Our observation suggests that the nonlinear dynamics of finite perturbations on a homogeneous steady state may be characteristically different from what is expected from a linear stability analysis.

The paper is organized as follows: In Sec. II we introduce Cattaneo's form of a general reaction-diffusion system with finite relaxation time of flux. In Sec. III we derive the equation of spatiotemporal perturbations around an unstable steady state assuming a cubic form of kinetics. The linear and nonlinear regimes are identified. The dynamics in the nonlinear regime corresponds to self-excited oscillations of a Lienard system. The temporal dynamics in the weak and strong excitation limits has been analyzed in Secs. IV and V, respectively. We conclude in Sec. VI with a brief discussion of our main results.

II. REACTION-DIFFUSION EQUATION WITH FINITE RELAXATION TIME

We begin with a macroscopic description of dynamics of a field variable $u(r,t)$, a function of space (r) and time (t) in terms of a reaction-diffusion system. The reaction-diffusion equation can be constructed phenomenologically from the continuity equation with a source term $f(u)$,

$$\frac{\partial u(r,t)}{\partial t} = -\frac{\partial J(r,t)}{\partial r} + f(u), \tag{2.1}$$

where $J(r,t)$ is the flux of $u(r,t)$ and Fick's law,

$$J(r,t) = -D\frac{\partial u(r,t)}{\partial r}, \tag{2.2}$$

to obtain

$$\frac{\partial u(r,t)}{\partial t} = D\frac{\partial^2 u(r,t)}{\partial r^2} + f(u). \tag{2.3}$$

Here D is the diffusion coefficient for the field variable. It is pertinent to note that standard diffusion equation suffers from a pathological drawback [21–24]. This is basically due to the lack of inertia of the Brownian particles. A close look at the well-known Gaussian solution of the diffusion equation with a point source at $r=0$ and $t=0$ implies that field variable $u(r,t)$ is nonzero even at very large r no matter how small t is. This means that the particles have infinite speed. Furthermore, the motion remains unpredicted even for a very small time. These difficulties may be removed by introducing the concept of relaxation time of the flux. Or, in other words, a simple generalization of the reaction-diffusion equation is to include the effect of finite memory transport with Cattaneo's modification of Fick's law in the form [4,11–16]

$$J(r,t+\tau) = -D\frac{\partial u(r,t)}{\partial r}. \tag{2.4}$$

Equation (2.4) implies that a concentration gradient at a time t causes a flux at a later time $(t+\tau)$, where τ is the delay time or relaxation time of the particles in adjusting one definite direction of motion. Making an expansion of J in Eq. (2.4) up to first order in τ in a Taylor series and differentiating the resulting equation with respect to r, one obtains

$$\frac{\partial J(r,t)}{\partial r} + \tau\frac{\partial^2 J(r,t)}{\partial r \partial t} = -D\frac{\partial^2 u(r,t)}{\partial r^2}. \tag{2.5}$$

Furthermore, differentiation of Eq. (2.1) with respect to t yields

$$\frac{\partial^2 u(r,t)}{\partial t^2} = -\frac{\partial^2 J(r,t)}{\partial r \partial t} + f'(u)\frac{\partial u(r,t)}{\partial t}. \tag{2.6}$$

Elimination of $J(r,t)$ from Eqs. (2.5) and (2.6) with the help of Eq. (2.1) results in the following reaction-telegraph equation, a hyperbolic equation of the form

$$\tau\frac{\partial^2 u(r,t)}{\partial t^2}+[1-\tau f'(u)]\frac{\partial u}{\partial t}=D\frac{\partial^2 u(r,t)}{\partial r^2}+f(u). \quad (2.7)$$

This equation imparts a dissipative character into the dynamics provided the dissipation constant (parentheses of $\frac{\partial u}{\partial t}$) is positive—i.e., $f'(u)<1/\tau$ for all u.

As $\tau\to 0$, the reaction-telegraph equation reduces to the usual parabolic differential equation—i.e., the ordinary reaction diffusion equation (2.1). Defining $\phi=1/\tau$, Eq. (2.7) may also be written in the form

$$\frac{\partial^2 u(r,t)}{\partial t^2}+[\phi-f'(u)]\frac{\partial u}{\partial t}=\phi D\frac{\partial^2 u(r,t)}{\partial r^2}+\phi f(u). \quad (2.8)$$

The reaction-diffusion equation with memory as above and its variants have been the subject of investigation by several authors. For example, propagating front solutions have been studied by Gallay and Raugel [27] for Eq. (2.7) without the $-\tau f'(u)$ term. Others [28,29] have studied the full reaction-diffusion equation for a finite speed of propagation. As pointed out earlier, Horsthemke [24] has investigated this equation for studying linear instabilities in reaction random walks with direction-independent kinetics. Attempts have also been made to analyze [26,30] delayed population growth models and in the related context of heat conduction and transport processes.

Equations (2.7) and (2.8) form the basis of our further analysis.

III. NONLINEAR DYNAMICS OF FINITE SPATIOTEMPORAL PERTURBATIONS

Our purpose in this section is to analyze the dynamics of spatiotemporal perturbations on a homogeneous steady state. Since the perturbation is finite, to make a nonlinear analysis of the problem, one requires that all the derivatives of the source term $f(u)$ and its derivative $f'(u)$ in Eq. (2.7) be around this steady state. A simple scheme of linearization around the steady state is untenable. In what follows we will consider a class of polynomials, particularly of cubic variety for the present purpose. The homogeneous steady states are the fixed points u_0 of the dynamical system defined as

$$f(u_0)=0. \quad (3.1)$$

The spatiotemporal perturbation $\delta u(r,t)$ on a homogeneous steady state is given by

$$u(r,t)=u_0+\delta u(r,t). \quad (3.2)$$

We then write the Taylor expansion of the following functions in terms of a finite series in perturbation:

$$f(u)=f(u_0)+f'(u_0)\delta u+\frac{f''(u_0)}{2!}\delta u^2+\frac{f'''(u_0)}{3!}\delta u^3+\cdots+\frac{f^n(u_0)}{n!}\delta u^n, \quad (3.3)$$

$$f'(u)=f'(u_0)+f''(u_0)\delta u+\frac{f'''(u_0)}{2!}\delta u^2+\cdots+\frac{f^n(u_0)}{(n-1)!}\delta u^{n-1}. \quad (3.4)$$

Making use of the expansions (3.3) and (3.4) in Eq. (2.7), we obtain

$$\frac{\partial^2}{\partial t^2}(\delta u)+\left[\phi-\left(f'(u_0)+\cdots+\frac{f^n(u_0)}{(n-1)!}(\delta u)^{n-1}\right)\right]\frac{\partial}{\partial t}(\delta u)=D\phi\frac{\partial^2}{\partial r^2}(\delta u)+\phi\left[f'(u_0)(\delta u)+\cdots+\frac{f^n(u_0)}{n!}(\delta u)^n\right]. \quad (3.5)$$

To proceed further we assume a reaction term of the cubic form

$$f(u)=p(u-a)-q(u-a)^3, \quad (3.6)$$

where p, q, and a are all positive constants. Our choice is guided by the fact that for an acceptable description of reacting and diffusing systems, the density $u(r,t)$ must preserve positivity for all $t\geqslant 0$ and all r for an appropriate choice of parameter space. Furthermore, cubic polynomials are well-known candidates [31–34] for a wide class of reactions in nonlinear chemical dynamics. Zeldovich and Frank-Kamenetsky [31] used cubic polynomials in a reaction-diffusion system to represent flame propagation as early as in 1938. In the 1940s cubic forms were considered for a model of impulse propagation along active nerve fiber, with $u(r,t)$ representing the voltage across the cell membrane [32]. The systematically designed algorithm for real chemical oscillator with two autocatalytic reactions, chlorite-iodide and arsenite-iodide reactions, is also based on cubic nonlinearity [33].

The homogeneous steady states u_0 of the dynamical system described by the kinetic term (3.6) are $u_0=a$, $a-\sqrt{p/q}$, and $a+\sqrt{p/q}$ of which $u_0=a$ is the linearly unstable fixed point. The derivatives $f^n(u_0)$ at $u_0=a$ are given by

$$f'(a)=p, \quad f''(a)=0, \quad f'''(a)=-6q. \quad (3.7)$$

The derivatives higher than third order are all zero for this case. Making use of the above relations in Eq. (3.5), we obtain the following nonlinear equation for evolution of spatiotemporal perturbation:

$$\frac{\partial^2}{\partial t^2}(\delta u)+[\phi-\{p-3q(\delta u)^2\}]\frac{\partial}{\partial t}(\delta u)=D\phi\frac{\partial^2}{\partial r^2}(\delta u)+p\phi(\delta u)-q\phi(\delta u)^3. \quad (3.8)$$

Let us now consider the spatiotemporal perturbation as a product of a space part and a time part of the form

$$\delta u(r,t)=\delta v(t)\cos(k\cdot r), \quad (3.9)$$

where k denotes the wave vector. Putting (3.9) into Eq. (3.8) and keeping harmonic balance on both sides, we obtain after a little bit of straightforward algebra

$$(\ddot{\delta v}) + (3q/2)[(\delta v)^2 - (2/3q)(p-\phi)](\dot{\delta v}) + \phi[Dk^2 - p](\delta v) + (\phi q/2)(\delta v)^3 = 0. \quad (3.10)$$

In carrying out a harmonic balance approximation, we have discarded the terms of spatial higher harmonics [using the relation $\cos^2(kr) = [\frac{1}{2} + \frac{1}{2}\cos(2kr)]$ from the nonlinear terms ∂u^2 and ∂u^3 in Eq. (3.8). Equation (3.10) is a key equation of this paper. It shows how a finite spatiotemporal perturbation (3.9) acts on a homogeneous unstable steady state and develops in time and space. The dynamics is essentially controlled by three parameters a, p, and q of the source function $f(u)$, delay time τ, and diffusion coefficient D. The equation takes care of nonlinearity of all orders. The interesting physical situations that might emerge both in the linear and nonlinear regimes are based on the relation between the relaxation time τ of the flux and the time scale $1/f'(u_0)$ of the reaction and the wavelength of the perturbation set by k. To explore the underlying nature of the dynamics, we now consider the following cases.

When the perturbation is infinitesimally small, the nonlinear terms $(\delta v)^2$ and $(\delta v)^3$ in Eq. (3.10) may be neglected. The equation reduces to

$$\ddot{\delta v} + (\phi - p)\dot{\delta v} + \phi[Dk^2 - p]\delta v = 0. \quad (3.11)$$

Equation (3.11) suggests that depending on the relative magnitude of ϕ, p, and Dk^2 the homogeneous unstable steady state is either stabilized by diffusion or retains the instability due to local activation of the reaction term. Thus, for a slow reaction rate—i.e., low p compared to the relaxation rate ϕ—the short wavelength perturbation (large k) decays in the long-time limit and the system returns to the homogeneous steady state.

The focal theme of the paper lies on the analysis of the nonlinear regime. To this end one retains the nonlinear terms $(\delta v)^2$ and $(\delta v)^3$ in Eq. (3.10). Under the condition $Dk^2 > p > \phi$, Eq. (3.10) assumes the form of a typical Lienard system [1] as given by

$$\ddot{X} + h(X)\dot{X} + g(X) = 0, \quad (3.12)$$

where the following abbreviations have been used:

$$X \equiv \delta v, \quad (3.13)$$

$$h(X) \equiv h(\delta v) = \frac{3q}{2}\left[(\delta v)^2 - \frac{2}{3q}(p-\phi)\right],$$

$$g(X) \equiv g(\delta v) = \phi(Dk^2 - p)(\delta v) + \frac{\phi q}{2}(\delta v)^3. \quad (3.14)$$

The forms of $h(X)$ and $g(X)$ satisfy the conditions for the Lienard theorem [1] to hold good. The special case of the Lienard system is a van der Pol oscillator for which the cubic term in $g(X)$ is absent. Thus, if (i) $g(X)$ is an odd function for all X and (ii) greater than zero $[g(X) > 0]$ for $X > 0$ and (iii) $h(X)$ is an even function for all X and (iv) odd function $F(X) = \int h(X')dX'$ has exactly one positive zero at $X = a$, $a = \sqrt{\frac{2(p-q)}{3q}}$, is negative for $0 < X < a$, is positive and nondecreasing for $X > a$ and $F(X) \to \infty$ as $X \to \infty$, then the dynamical system admits of an unique stable limit cycle surrounding the origin in the phase plane.

The above discussion clearly leads to a conclusion that in the range $Dk^2 > p > \phi$, the finite spatiotemporal perturbation (3.9) neither grows nor decays, but goes over to self-excited periodic oscillations following Eq. (3.10). Or, in other words, the spatial pattern pulsates over a regular interval of time dictated by the time period (T) of the limit cycle such that $\delta v(t) = \delta v(t+T)$. Finally, the form of the spatiotemporal perturbation (3.9) must satisfy the appropriate boundary condition. For example, an application of the zero-flux boundary condition $\frac{\partial(\delta u(r,t))}{\partial r}|_{0,L} = 0$ on (3.9) yields a discrete set of wave numbers $k_n = n\pi/L$, where L is the length of the reaction medium, n being a nonzero integer.

Note that Eq. (3.10) cannot be solved exactly for $\delta v(t)$. In what follows we present two approximate schemes for the analysis of the dynamics of weak and strong excitations in the next section for calculation of the time period T.

IV. TEMPORAL DYNAMICS IN THE WEAK EXCITATION LIMIT

We now return to Eq. (3.10) and rewrite it in the form

$$\ddot{\delta v} + \alpha(\delta v) = -\epsilon[(\delta v)^2 - \mu]\dot{\delta v} - \beta(\delta v)^3, \quad (4.1)$$

with the abbreviations

$$\alpha = \phi(Dk^2 - p), \quad \epsilon = \frac{3q}{2}, \quad \beta = \frac{\phi q}{2}, \quad \mu = \frac{2(p-\phi)}{3q}. \quad (4.2)$$

Our object in this section is to find out the approximate amplitude and frequency and an approximate solution of Eq. (4.1) (when the nonlinear excitation ϵ is finite but small) by employing the harmonic balance method. For the sake of completeness, we give a brief recipe of the technique involved [35].

To simulate an asymptotic periodic orbit in the spirit of a limit cycle solution of Eq. (4.1), we assume

$$\delta v = A\cos(\omega t), \quad (4.3)$$

where A and ω are the amplitude and the frequency to be determined.

Putting (4.3) into Eq. (4.1), we obtain after a little algebra

$$(-\omega^2 + \alpha)A\cos(\omega t) = \epsilon\omega\left(\frac{A^2}{4} - \mu\right)A\sin(\omega t) - \left(\frac{3\beta A^2}{4}\right)A\cos(\omega t) + \left(\frac{\epsilon\omega A^2}{4}\right)A\sin(3\omega t) - \left(\frac{\beta A^2}{4}\right)\cos(3\omega t). \quad (4.4)$$

Neglecting higher harmonics in the weak nonlinear limit and matching the coefficients of sine and cosine terms, we obtain

$$\omega^2 = \alpha + \frac{3\beta A^2}{4},$$

$$A^2 = 4\mu. \tag{4.5}$$

The dependence of the frequency on amplitude in (4.5) is a typical hallmark of nonlinear oscillations.

To obtain the approximate solution of the equation, the harmonic balance method can be used for pseudolinearization of the nonlinear equation. The idea is to find out a linear substitute for the right-hand side of (4.1). Putting (4.3) on the right-hand side of Eq. (4.1) and discarding the higher harmonics, one obtains

$$\ddot{\delta v} + \alpha \delta v \simeq \epsilon\omega\left(\frac{A^2}{4} - \mu\right)[A\sin(\omega t)] - \left(\frac{3\beta A^2}{4}\right)[A\cos(\omega t)]. \tag{4.6}$$

Substituting the cosine and sine terms in Eq. (4.6) by $\delta v(t)$ and its derivative, respectively, we arrive at the following linear equation after a little bit of rearrangement:

$$\ddot{\delta v} + \left(\alpha + \frac{3\beta A^2}{4}\right)\delta v + \epsilon\left(\frac{A^2}{4} - \mu\right)\dot{\delta v} = 0. \tag{4.7}$$

The above equation may be rewritten in the form

$$\ddot{\delta v} + B\dot{\delta v} + C\delta v = 0, \tag{4.8}$$

where

$$B = \frac{3q}{2}\left[\frac{A^2}{4} - \frac{2(p-\phi)}{3q}\right],$$

$$C = \phi\left[Dk^2 - p + \frac{3qA^2}{8}\right]. \tag{4.9}$$

With the initial condition $\delta v(t=0)=A$ and $\delta v(t=0)=0$, we obtain the solution of Eq. (4.8):

$$\delta v(t) = A\exp\left(-\frac{Bt}{2}\right)\cos\left[\sqrt{C - \frac{B^2}{4}}(t)\right]. \tag{4.10}$$

The condition for existence of the limit cycle implies $B=0$—i.e., $A^2=8(p-\phi)/3q$ [see the second equation of (4.5)]—and the time period in the weak excitation limit is determined by $\sqrt{C}$.

The approximate analytical form of the weak but finite spatiotemporal perturbation for the nth mode under the zero-flux boundary condition can therefore be found from Eqs. (3.9) and (4.10) as

$$\delta u(r,t) = A\cos\left[\sqrt{\left[\phi\left(Dk_n^2 - p + \frac{3qA^2}{8}\right)\right]}(t)\right]\cos(k_n r). \tag{4.11}$$

In order to illustrate the spatiotemporal behavior of the dynamics of perturbations, we extend the above analysis to two dimension. In this case the analytical expression (4.11) becomes

$$\delta u(x,y,t) = A\cos(\sqrt{C}t)\cos(k_x x)\cos(k_y y), \tag{4.12}$$

where $C=\phi(Dk_n^2 - p + 3qA^2/16)$ and $k_n^2 = k_x^2 + k_y^2$, with $k_x = n_x\pi/L_x$ and $k_y = n_y\pi/L_y$.

Here n_x and n_y are integers 1, 2, 3… specifying the number of nodes in two directions. The change in numerical factor in C in (4.12) is due to the harmonic balance approximation carried out to obtain Eq. (3.10) in two dimensions where it is necessary to take care of the expressions $\cos^2 k_x x = (\frac{1}{2} + \frac{1}{2}\cos 2k_x x)$ and $\cos^2 k_y y = (\frac{1}{2} + \frac{1}{2}\cos 2k_y y)$ in discarding higher harmonics. To check the analytical expression (4.12), we numerically integrate Eq. (4.1) directly for a representative set of parameters values, $p=1.0$, $q=0.5$, $\phi=0.5$, $D=0.2$, and $L_x=L_y=1.0$, to obtain $\delta v(t)$. Multiplying this quantity with the spatial contribution $\cos k_x x \cos k_y y$ yields $\delta u(x,y,t)$. We depict in Figs. 1 and 2, the sequence of patterns at definite time interval for $n_x=n_y=2$ and $n_x=4$, $n_y=2$, respestively. This matches fairly well with $t_m = \frac{m\pi}{2\sqrt{C}}$ (with m being an integer) corresponding to the interval determined by C in Eq. (4.12). This collapse and revival of spatial patterns is clearly a result of instability of homogeneous steady state and is transient in nature. The conspicuous appearance of relaxation time $\tau\,(=1/\phi)$ of the diffusive flux as a multiplying factor in the time period indicates that the time scale of spatiotemporal dynamics is set by the relaxation behavior of the reaction-diffusion system.

V. DYNAMICS IN THE STRONG EXCITATION LIMIT

This section is devoted to the discussion on strong nonlinear limit (i.e., when the nonlinear excitation ϵ is finite and large). Rewriting Eq. (4.1) as

$$\ddot{X} + \alpha X + \beta X^3 + \epsilon[X^2 - \mu]\dot{X} = 0, \tag{5.1}$$

where $X \equiv \delta v$, and defining

$$F(X) = \frac{X^3}{3} - \mu X$$

and

$$W = \dot{X} + \epsilon F(X), \tag{5.2}$$

Eq. (5.1) can be expressed as

$$\dot{X} = W - \epsilon F(X),$$

$$\dot{W} = -\alpha X - \beta X^3. \tag{5.3}$$

Furthermore, with $Y = W/\epsilon$, Eq. (5.3) is equivalent to

$$\dot{X} = \epsilon[Y - F(X)],$$

$$\dot{Y} = \frac{1}{\epsilon}[-\alpha X - \beta X^3]. \tag{5.4}$$

The above equations therefore clearly reveal two distinct time scales in the dynamics when ϵ is large. Following standard procedure [1], it is easy to show that the motion along the nullcline $Y=F(X)$ following the Y direction is slow compared to the motion along the X direction in the Lienard plane (X,Y). The dynamical system exhibits relaxation oscillation [1,35]. The total time period of oscillations or limit cycle is typically the total time spent on the slower branches

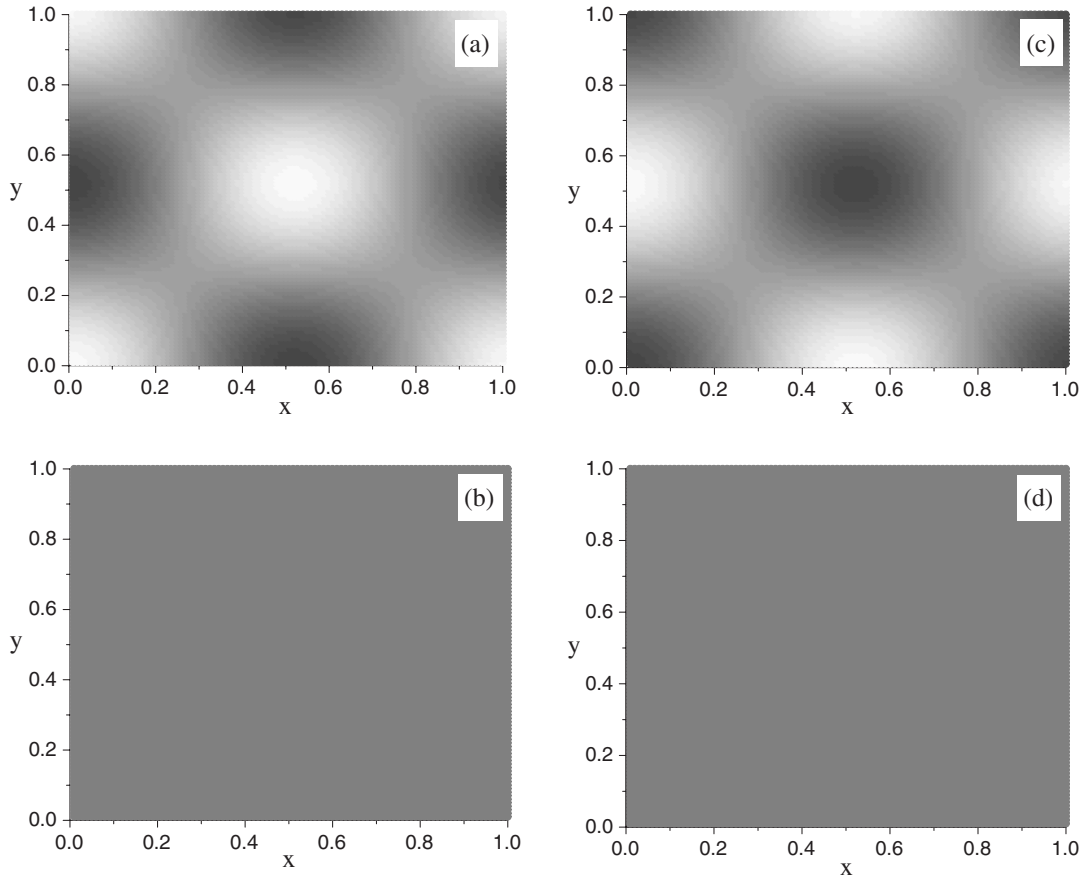

FIG. 1. Evolution of a spatial pattern in two dimensions for the set of parameter values (as mentioned in the text) at different times for (2×2) nodes: (a) time=0, (b) time=$(\pi/2\sqrt{C})$, (c) time=$(\pi/\sqrt{C})$, and (d) time=$(3\pi/2\sqrt{C})$.

of the nullcline $Y=F(X)$, which can be calculated as

$$T=\frac{1}{\phi}\left\{\left[3+\left(\frac{p-\phi}{Dk^2-p}\right)\right]\ln\left(\frac{Dk^2-\frac{2\phi}{3}-\frac{p}{3}}{Dk^2-\frac{\phi}{3}-\frac{2p}{3}}\right)+\left(\frac{2(p-\phi)}{Dk^2-p}\right)\ln 2\right\}. \quad (5.5)$$

It is apparent that in the limit of large nonlinearity, the time period is again a function of reaction rate and relaxation rate of the flux.

VI. CONCLUSION

Based on Cattaneo's modification of the reaction-diffusion equation which takes care of inertia of the Brownian particles undergoing diffusion in terms of finite relaxation time of flux and a cubic reaction term, we have analyzed the nonlinear dynamics of finite spatiotemporal perturbations of a homogeneous steady state. It has been shown that the reaction time, relaxation time of the flux, and diffusion coefficient are the guiding factors in addition to the nonlinearity of the reaction term in determining the dynamics.

The model at the microscopic level takes care of the persistent stochastic motion ranging from the ballistic to the diffusive regime. While the earlier linear analysis has clearly established that for the one-variable model a small wavelength perturbation exhibits damped oscillations for a large reaction time where the transient or oscillatory character of the dynamics owes its origin to the ballistic nature or short, finite relaxation time of the diffusing flux, the present nonlinear analysis of perturbations on a homogeneous steady state, on the other hand, has demonstrated a new spatiotemporal instability. This bears the imprint of relaxation other than the aforesaid guiding factors. Our observations may be summarized as follows.

(i) The nonlinear contribution of the reaction term yields a temporal dynamics which is a result of instability of the homogeneous state leading to self-excited oscillations of a Lienard system. The origin of the oscillations can be traced to an interplay between the small, finite relaxation time of the diffusing flux and the time scale of the reaction. This leads to a collapse and revival of spatial patterns of definite wavelength at time intervals determined by the time period of the underlying limit cycle.

(ii) The homogeneous steady state of a reaction-diffusion system can undergo diffusion-driven Turing instability initiating stationary pattern formation as a result of the competi-

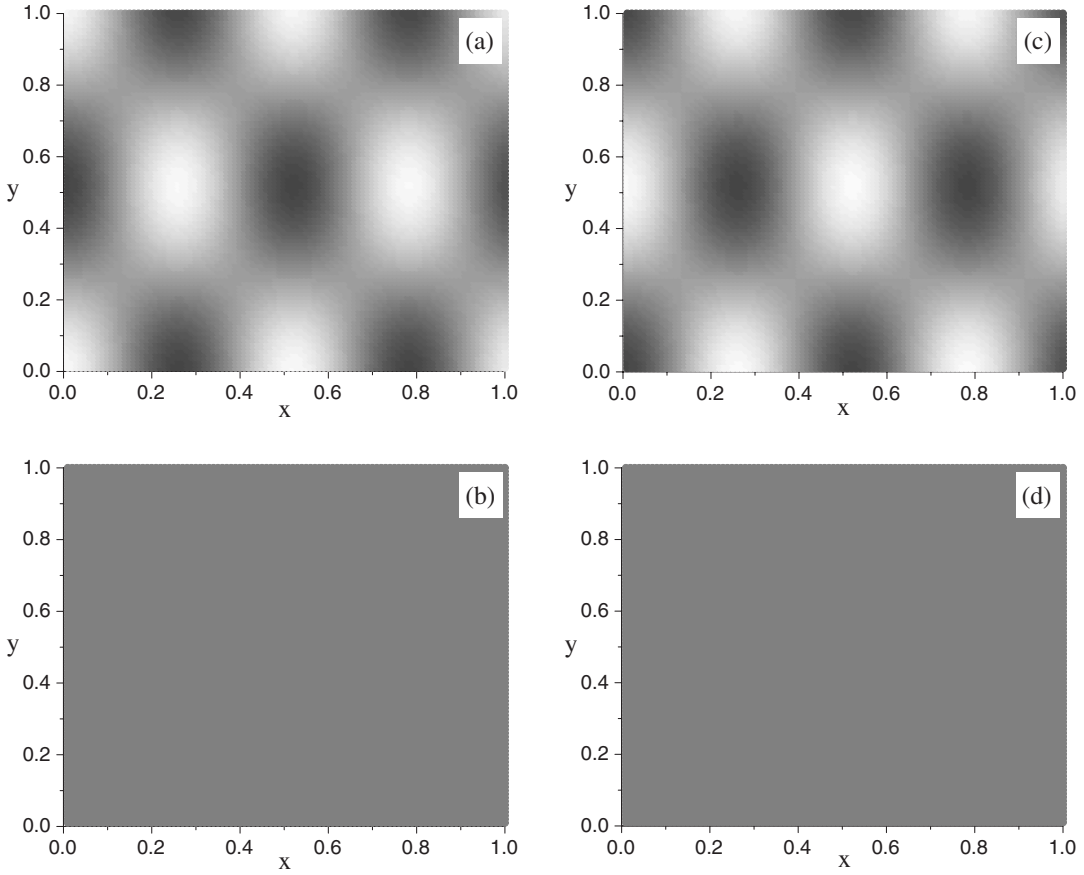

FIG. 2. Evolution of a spatial pattern in two dimensions for the set of parameter values as in Fig. 1 at different times for (4×2) nodes: (a) time=0, (b) time=$(\pi/2\sqrt{C})$, (c) time=$(\pi/\sqrt{C})$, and (d) time=$3(\pi/2\sqrt{C})$.

tion between short-range activation and long-range inhibition for a minimum of two variables (activator and inhibitor). The transient spatial pattern in the form of collapse and revival, on the other hand, occurs for a one-variable system and is primarily relaxation driven rather than diffusion driven in its origin and therefore differs from Turing instability.

(iii) A discussion of the relative role played by the diffusion coefficient and relaxation time of the diffusing flux may be in order. A close look at the expressions for the time periods of oscillations clearly reveals that the relaxation rate appears as a significant multiplying factor in all of them. This implies that just as the oscillating time scales for short-wavelength modes result from the relaxation behavior of the flux as evident from linear analysis, the time scale of nonlinear self-excited oscillations in our analysis is set by the relaxation time in addition to the strength of the nonlinearity of the reaction term. The nonlinear dynamics of finite perturbations on a homogeneous steady state therefore differs generically from what we observe from linear stability analysis.

Before closing this section it is pertinent to note that the success of the nonlinear analysis performed here is strongly related to the fact that higher-order derivatives (higher than third) vanish in the expansion (3.5). The cubic polynomial serves as an well-known paradigm in autocatalytic chemical reactions, the Hodgkin-Huxley model, the flame propagation problem, etc. The scope for a generalization of the method, however, is somehow limited by the mathematical difficulties in tackling higher-order perturbation times on an equal footing, except in some specific cases of the polynomial source term and appropriately chosen homogeneous steady states of the systems. We hope that suitable perturbation techniques with good convergence strategies may be useful for further advancement of the theory of nonlinear stability analysis in search of new dynamical and spatiotemporal instabilities.

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[1] See, for example, S. H. Strogatz, *Nonlinear Dynamics and Chaos* (Addison-Wesley, Reading, MA, 1994).
[2] J. Garcia-Ojalvo and J. M. Sancho, *Noise in Spatially Extended Systems* (Springer, NewYork, 1999).
[3] N. F. Britton, *Reaction-diffusion Equations and Their Application to Biology* (Academic, New York, 1986).
[4] A. Okubo, *Diffusion and Ecological Problems* (Spinger, Berlin, 1980).
[5] I. R. Epstein and J. A. Pojman, *An Introduction to Nonlinear Chemical Dynamics: Oscillations, Waves, Patterns and Chaos* (Oxford University Press, New York, 1998).
[6] P. Gray and S. K. Scott, *Chemical Oscillations and Instabilities* (Clarendon Press, Oxford, 1990).
[7] A. M. Turing, Philos. Trans. R. Soc. London, Ser. B **237**, 37 (1952).
[8] P. De Kepper, J. Boissonade, and I. R. Epstein, J. Phys. Chem. **94**, 6525 (1990).
[9] *Oscillations and Travelling Waves in Chemical Systems*, edited by R. J. Feld and M. Burger (Wiley, NewYork, 1985).
[10] H. Meinhardt, *Models of Biological Pattern Formation* (Academic, London, 1982).
[11] A. Becker and L. Kramer, Phys. Rev. Lett. **73**, 955 (1994).
[12] C. Van den Broeck, J. M. R. Parrondo, J. Armero, and A. Hernandez-Machado, Phys. Rev. E **49**, 2639 (1994).
[13] P. S. Landa, A. A. Zaikin, and L. Schimansky-Geier, Chaos, Solitons Fractals **9**, 1367 (1998).
[14] S. Dutta, S. S. Riaz and D. S. Ray, Phys. Rev. E **71**, 036216 (2005).
[15] S. S. Riaz, S. Dutta, S. Kar, and D. S. Ray, Eur. Phys. J. B **47**, 255 (2005).
[16] S. S. Riaz, R. Sharma, S. P. Bhattacharyya, and D. S. Ray, J. Chem. Phys. **127**, 064503 (2007).
[17] U. Dressler and J. D. Farmer, Physica D **59**, 365 (1992).
[18] E. Aurell, B. Boffetta, A. Crisanti, G. Paladin, and A. Valpiani, Phys. Rev. Lett. **77**, 1262 (1996).
[19] A. Torcini, P. Grassberger, and A. Politi, J. Phys. A **28**, 4533 (1995).
[20] M. Cencini and A. Torcini, Phys. Rev. E **63**, 056201 (2001).
[21] C. Cattaneo, C. R. Acad. Sci. URSS **247**, 431 (1958).
[22] K. P. Hadeler, in *Reaction Transport Systems in Mathematics Inspired by Biology*, edited by V. Capasso and O. Dickmann (Springer-Verlag, Berlin, 1998).
[23] T. Hillen, Math. Methods Appl. Sci. **8**, 507 (1998).
[24] W. Horsthemke, Phys. Rev. E **60**, 2651 (1999).
[25] S. Kar, S. K. Banik, and D. S. Ray, Phys. Rev. E **65**, 061909 (2002).
[26] S. Kar, S. K. Banik, and D. S. Ray, J. Phys. A **36**, 2771 (2003).
[27] Th. Gallay and G. Raugel, ZAMP **48**, 451 (1997); J. Differ. Equations **150**, 42 (1998).
[28] K. P. Hadeler, Cad. Appl. Math. Qu. **2**, 27 (1994).
[29] V. Mendez and A. Compte, Physica A **260**, 90 (1998).
[30] K. K. Manne, A. J. Hurd, and V. M. Kenkre, Phys. Rev. E **61**, 4177 (2000); S. Fedotov, Phys. Rev. Lett. **86**, 926 (2001); J. M. Sancho and A. Sanchez, Phys. Rev. E **63**, 056608 (2001).
[31] Ya. B. Zeldovich and D. A. Frank-Kamenetsky, Dokl. Akad. Nauk SSSR **19**, 693 (1938); Ya. B. Zeldovich and G. I. Darrenblalt, Combust. Flame **3**, 61 (1959).
[32] F. Offner, A. Weinberg, and C. Young, Bull. Math. Biophys. **2**, 89 (1940).
[33] P. de Kepper, I. R. Epstein, and K. Kautin, J. Am. Chem. Soc. **103**, 6121 (1981).
[34] S. Dutta and D. S. Ray, Phys. Rev. E **75**, 066206 (2007); **77**, 036202 (2008).
[35] J. K. Bhattacharjee, A. K. Mallik, and S. Chakraborty, Indian J. Phys., A **81**, 11 (2007).